

---

# Fluid Dynamical Descriptions

BJORN STEVENS AND PIER SIEBESMA

From a fluid-dynamical point of view clouds are a turbulent dispersion of condensate in a multi-phase and multi-component flow. Their description thus evokes the language of fluid mechanics. The necessarily multi-phase fluid-mechanical description of cloudy flows thus requires a more sophisticated understanding of thermodynamics, then for instance one would encounter in single-component and single-phase flows. In this chapter we present a concise overview of the main equation systems, and concepts, that are used to describe clouds and cloudy flows. It is assumed that the reader has a basic background in thermodynamics and fluid mechanics, and this text attempts to build on this background to introduce the special elements related to both topics as applied to potentially cloudy systems.

## 4.1 Thermodynamic Systems

### 4.1.1 A Multi-Component Multi-Phase System

The atmosphere, or air, as we experience it, is a multi-component gas in which a great variety (if not great amount) of finescale particulate matter is suspended. The gas phase constituents include several major gases (Nitrogen, Oxygen, Argon) which through the current era have existed in a relatively fixed proportion to one another. To a large degree these determine the thermodynamic properties of “dry air”, that is an ideal mixture composed of 78.08 % N<sub>2</sub>, 29.05 % O<sub>2</sub> and 0.934 % Ar. Additionally, the atmosphere contains variable vapors such as carbon dioxide and water, along with a host of seemingly minor gases (e.g., Neon, Helium, Nitrous Oxide, Ozone, Methane and other organics) some of which can be important for determining the radiative properties of the atmosphere and the quality of the air we breath. Of the variable constituent, water is the most striking as it ranges from abundances that vary over many many orders of magnitude, from nearly zero in the coldest regions of the upper troposphere, to as much as 4% by volume over very warm bodies of water. Because of its proclivity to change phase and the manner in which these phase changes affect the local temperature on the one hand, and foster diverse interactions with radiant energy on the other, water indelibly marks motions in the lower atmosphere on all time and spatial scales so that it is hard to think properly about atmospheric motions, let along clouds, without considering how water is coupled to them. In this sense the simplest, accurate

description of the dynamic atmosphere demands requires a description that admits for at least two-components, dry air and water, with one component (water) admitting multiple phases.

The basic thermodynamic properties of the atmosphere thus depend on its component parts. Typically these component parts are defined in terms of their mass,  $m$  such that for an equilibrium system four constituents of the moist atmosphere can be defined, dry air, vapor, liquid water, and solid (ice) water, denoted by subscripts d, v, w and i respectively. The total mass of the system is thus given by sum of the constituent masses such that

$$M = M_d + M_v + M_w + M_i. \quad (4.1)$$

The normalized, or specific mass of a component  $x$  is denoted by  $q_x = M_x/M$ . Is useful to distinguish between equilibrium condensed phases associated with clouds, which evolve with the thermodynamic state in a more or less irreversible way, and non-equilibrium phases, generally larger hydrometeors that develop through irreversible microphysical processes such as the collision and coalescence of water droplets. Because they are generally larger, non-equilibrium phases of water in the atmosphere are more dilute and short-lived. Their presence requires the introduction of a more expansive view of the thermodynamic constituents within a moist atmosphere, for instance, the mass of rain, snow, graupel and hail. The focus in this section is on a thermodynamic description for an equilibrium system.

Local thermodynamic systems are identified with control volumes, sometimes referred to as air parcels, which are assumed to be much smaller than the scale over which thermodynamic properties vary. Diffusion rapidly homogenizes the atmosphere on scales smaller than the Kolmogorov length scale,  $\eta = (\nu^3/\varepsilon)^{1/4}$ , where  $\nu$  is the viscosity of the atmosphere and  $\varepsilon$  is the turbulent dissipation rate. In vigorous cumulus clouds the dissipation rate may approach  $0.05 \text{ m}^2 \text{ s}^{-3}$ , which given a kinematic viscosity,  $\nu = 1.5 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$  implies that variations in thermal properties are not present on scales less than  $\eta = 0.5 \text{ mm}$ . This is several thousand times larger than the mean free path of an air molecule, making the concept of an air parcel a useful one.

A special simplification of atmospheric thermodynamics arises from the very small volume fraction of the condensate phases. Typically the specific condensate mass within a cloud is less than  $1 \text{ g kg}^{-1}$ , and given the approximately thousand fold increase in density in the condensate versus vapor phases (for typical atmospheric pressures) this implies

that the volume fraction of condensate in the atmosphere can safely be assumed to be negligibly small. The dry-air and vapor are further approximated as ideal gases (point particles) in an ideal mixture, so that the total volume of the system is seen by both the gas and vapor constituents, i.e.,  $\alpha_v = V/M_v$  denotes the specific volume of the vapor. Because the population of condensate particles is also relatively dilute, the concept of an equilibrium thermodynamic system in the presence of condensate often needs to be relaxed to approximately describe the state of larger systems, ones in which a large population of condensate particles can be assumed to be randomly distributed. In this case, one often imagines an air parcel on the scale of a  $1 \text{ m}^3$ . Strictly speaking volumes of air this large cannot be thought of in terms of a single temperature, but the error of this approximation is typically much less than those associated with other approximations invoked in the description of such systems.

#### 4.1.2 A notational challenge

A particular challenge of describing moist atmospheric systems is notational. Many symbols are overloaded. As an example, the symbol  $v$  is used to denote the specific volume in thermodynamic systems, or the second component of the velocity vector in fluid dynamical systems. The roman form, “v” of the same letter is used to denote, in the form of a subscript “vapor”, and sometimes “virtual”. As a further example,  $s$  is often used to denote entropy, but also dry static energy. Its roman form denotes saturation, or a surface quantity. This chapter is not the proper place to attempt to systematically overhaul an antiquated terminology, but some slight deviations are introduced. For instance we refrain from the use of the virtual temperature terminology in favor of a density temperatures, and we use fraktur fonts for classical thermodynamical functions such as entropy,  $\mathfrak{s}$ , enthalpy  $\mathfrak{h}$  and the gibbs energy  $\mathfrak{g}$ ; and  $\mathfrak{q}$  is used to denote heating so as to distinguish it from,  $q$ , the specific mass. Implicit above is also the distinction between roman fonts which reference a word, such as “w” for liquid water, and italic fonts for mathematical variables, e.g.,  $w$  for the vertical velocity. The value of this distinction is not simply to facilitate the further overloading of symbols, but to distinguish, for instance, “max” from *max* the former denoting a maximum, the latter a product of three variables,  $m$ ,  $a$  and  $x$ .

#### 4.1.3 Equation of State

Taking an air-parcel to be comprised of an ideal mixture of ideal gases, perhaps in the presence of condensate, the equation of state is that for an ideal gas of variable composition, such that

$$p = p_d + p_v = \left( \frac{M_d R_d}{V} + \frac{M_v R_v}{V} \right) T, \quad (4.2)$$

where  $R_d$  and  $R_v$  are the specific gas constants of ‘dry air’ and water vapor respectively. Introducing subscript ‘c’ and subscript ‘t’ to denote the total amount of water, and total

amount of condensate respectively, so that

$$q_t = q_v + q_w + q_i = q_v + q_c, \quad (4.3)$$

and defining the density of the gaseous/vapor mixture as  $\rho = (M_d + M_v)$  allows one to formulate the equation of state as

$$p = \rho R T \quad (4.4)$$

where the specific gas constant depends on the amount and distribution of water,

$$R = (1 - q_c)R_d + q_v(R_v - R_d). \quad (4.5)$$

To avoid dealing with a state dependent gas constant it is customary to define a density temperature, such that

$$p = \rho R_d T_\rho \quad \text{where} \quad T_\rho = T(1 + \epsilon q_v - q_c) \quad (4.6)$$

and

$$\epsilon = \left( \frac{R_v}{R_d} - 1 \right) \approx 0.608. \quad (4.7)$$

In the absence of water the density temperature is the air temperature, otherwise it can be interpreted as the temperature of a dry air parcel having the same density and temperature as the given air parcel.

Local density perturbations are associated with vertical accelerations, or the buoyancy,  $b$ , of an air parcel, whereby

$$b = -g \frac{\rho'}{\rho_0} \approx g \frac{T'_\rho}{T_{\rho,0}}. \quad (4.8)$$

from the definition of the density temperature, assuming that pressure perturbations are small compared to density perturbations, i.e.,  $p'/p_0 \ll T'/T_0$ .

#### 4.1.4 The First Law and its Consequences

For an atmospheric system it proves useful to use temperate and pressure as the thermodynamic coordinates. The choice of pressure, rather than volume as is more customary in the description of laboratory systems, is motivated by the fact that the atmosphere is an open system, so that pressure is fixed externally through the mass of air and the gravitational acceleration. In this form the first law is most usefully expressed as

$$\mathfrak{q} = d\mathfrak{h} - \alpha dp \quad (4.9)$$

where  $q$  is the heating, and  $h$  is the specific enthalpy, or heat function,

$$\mathfrak{h} = q_d \mathfrak{h}_d + q_v \mathfrak{h}_v + q_w \mathfrak{h}_w + q_i \mathfrak{h}_i. \quad (4.10)$$

The isobaric specific heat,  $c_p \equiv (\partial \mathfrak{h} / \partial T)_p$ , so that the enthalpy can be written as

$$\mathfrak{h} = (q_d c_{p,d} + q_v c_{p,v} + q_w c_w + q_i c_i) T, \quad (4.11)$$

which expresses the fact that for an ideal gas  $(\partial \mathfrak{h} / \partial p)_T$  vanishes, and the subscript  $p$  for the specific heats of liquid water and ice are neglected because these are assumed to be incompressible phases. The vaporization enthalpy,  $L_v$  is defined as the difference between the vapor and liquid water enthalpies, such that

$$L_v = \mathfrak{h}_v - \mathfrak{h}_w. \quad (4.12)$$

Table 4.1. Common thermodynamic constants

Thermodynamic Quantity	Value	Unit
$R_d$	0.2871	$\text{kJ kg}^{-1} \text{K}^{-1}$
$R_v$	0.4615	"
$c_{p,d}$	1.004	"
$c_{p,v}$	1.864	"
$c_w$	4.184	"
$c_i$	2.108	"
$L_v$	2500.8	$\text{kJ kg}^{-1}$
$L_s$	2834.1	"

Here the enthalpies of vaporization and sublimation are specified at 273.15 K, both decrease as temperature increases.

For an ice-free system one can use the definition of  $L_v$  to substitute for  $h_v$  in Eq. (4.10) such that

$$h_e = c_e T + q_v L_v \quad \text{where} \quad c_e = q_d c_{p,d} + q_t c_w. \quad (4.13)$$

Alternatively substituting for  $h_w$  in Eq. (4.10) yields

$$h_\ell = c_\ell T - q_w L_v \quad \text{where} \quad c_\ell = q_d c_{p,d} + q_t c_v. \quad (4.14)$$

The apparent differences in the enthalpies defined in Eq. (4.13) as compared to Eq. (4.14) arise from differences in the definition of the specific heats. Similar formulations based on the sublimation enthalpy can be developed for liquid-free systems. Values for the specific heats and the phase-change enthalpies are provided along with those for the gas constants in Table 4.5.

#### 4.1.5 The Second Law and its Consequences

The second law postulates the existence of an entropy function, denoted by  $S$ , such that in equilibrium values of other quantities maximize the entropy function. For a reversible process

$$T d\mathfrak{s} = d\mathfrak{h} - \alpha dp \quad (4.15)$$

As an extensive state function, like enthalpy, the entropy can be decomposed into its constituent parts, so that

$$\mathfrak{s} = q_d \mathfrak{s}_d + q_v \mathfrak{s}_v + q_w \mathfrak{s}_w + q_i \mathfrak{s}_i. \quad (4.16)$$

following Eq. (4.10). For an ideal gas, such as dry air, the entropy can be written in terms of a reference entropy so that for instance,

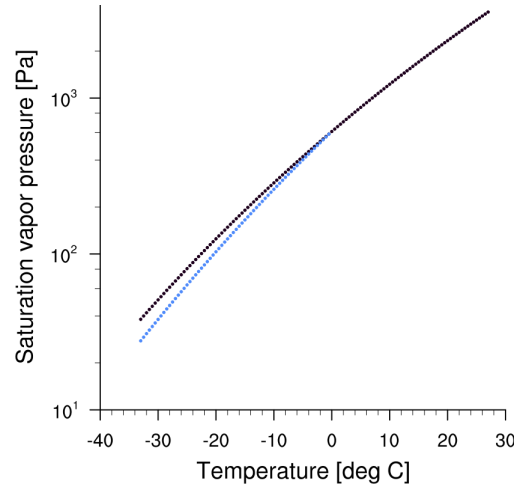
$$\mathfrak{s}_d = \mathfrak{s}_{d,0} + c_{p,d} \ln(T/T_0) - R_d \ln(p/p_0), \quad (4.17)$$

where  $\mathfrak{s}_0$  is the entropy of a reference state at temperature  $T_0$  and pressure  $p_0$ .

For an isobaric isothermal system it follows from Eq. (4.15) that for a reversible process

$$0 = d(\mathfrak{h} - T\mathfrak{s}), \quad (4.18)$$

which defines the Gibbs free energy,  $\mathfrak{g} = h - T\mathfrak{s}$ , i.e., the energy available to do work in an isothermal and isobaric system. Because, in equilibrium,  $h$  and  $T$  maximize  $\mathfrak{s}$ , the Gibbs potential of a system in equilibrium is a minimum.



**Figure 4.1** Saturation vapor pressure over liquid (dark grey) and ice (blue). At  $T = 0^\circ\text{C}$  the saturation vapor pressure is 610.15 Pa. At  $T = -30^\circ\text{C}$  the saturation vapor pressure over liquid water is 50.8 Pa as compared to 38.0 Pa over ice at the same temperature. Saturation with respect to liquid for  $T < 0^\circ\text{C}$  are relevant because super-cooled water is often present in the atmosphere, with homogeneous nucleation of ice particles first occurring at about  $T = -38^\circ\text{C}$

#### 4.1.6 The Clapeyron Equation

The condition that for an equilibrium system the Gibbs free energy is a minimum implies that for a system in which there is an equilibrium between two phases, say liquid water and water vapor, then the specific Gibbs energy of each phase must be equal, i.e.,  $\mathfrak{g}_v = \mathfrak{g}_w$ . Otherwise a redistribution of the mass between the phases could lower the total Gibbs energy. We further note that a change in the equilibrium state of the system, say associated with a change in temperature, implies a change in the Gibbs energy, such that

$$d\mathfrak{g} = d\mathfrak{h} - \mathfrak{s}dT - \alpha dp = \alpha dp - \mathfrak{s}dT, \quad (4.19)$$

hence

$$d\mathfrak{g}_v = \alpha_v dp - \mathfrak{s}_v dT \quad \text{and} \quad d\mathfrak{g}_w = \alpha_w dp - \mathfrak{s}_w dT. \quad (4.20)$$

But because the maintenance of equilibrium requires that  $d\mathfrak{g}_v = d\mathfrak{g}_w$  it follows that for such a transformation the vapor pressure changes with temperature as,

$$dp = \frac{\mathfrak{s}_v - \mathfrak{s}_w}{\alpha_v - \alpha_w} dT. \quad (4.21)$$

This is the Clapeyron equation describing how vapor pressure changes with temperature. By substituting for  $\alpha_v$  from the ideal gas law, and noting that for a saturated system  $\alpha_v \gg \alpha_w$  and that  $L_v/T = \mathfrak{s}_v - \mathfrak{s}_w$ , Eq. 4.21 can be written in the form

$$d(\ln p) = \frac{L_v}{R_v T^2} dT. \quad (4.22)$$

These approximations are due to Clausius and Eq. 4.22 has come to be called the Clausius Clapeyron Equation, Fig. ??.

### 4.1.7 Potential Temperatures

The first and second laws dictate how temperature changes between a give state, and a reference state defined by its pressure and phase distribution, i.e., the triplet  $\{p, q_v, q_w\}$ , which in the generic case is denoted by subscript  $\star$ . The reference state temperature,  $\theta_\star$ , is a property of the system and is often used to label the state of the parcel, as the reference state value of temperature is invariant for reversible transformations of the air parcel, and is thus a convenient way to compare air parcels in different parts of the atmosphere, where for instance the pressure or humidity may vary.

To illustrate the concept, consider a reference state where all of the vapor is condensed into liquid, such that  $q_{v,\star} = 0$  and  $p_\star = \pi = 10^5$  Pa. This particular reference state  $\{\pi, q_v, q_w\}_e = \{\pi, 0, q_t\}$  is denoted by subscript “e”, and the temperature attained by the air parcel when reversibly transformed to this state is denoted by  $\theta_e$ . For such a process,  $\mathfrak{s} = \mathfrak{s}_e$ , hence

$$q_d \mathfrak{s}_d + q_v \mathfrak{s}_v + q_w \mathfrak{s}_w - q_d \mathfrak{s}_{d,e} - q_t \mathfrak{s}_{w,e} = 0, \quad (4.23)$$

equivalently

$$q_d \mathfrak{s}_d + q_t \mathfrak{s}_w - q_d \mathfrak{s}_{d,e} - q_t \mathfrak{s}_{w,e} + q_v (\mathfrak{s}_v - \mathfrak{s}_w) = 0. \quad (4.24)$$

Without loss of generality an expressions for the entropy of the given state can be formed in terms of the values of the entropy in the reference state, from Eq. (4.17), such that

$$\mathfrak{s}_d = \mathfrak{s}_{d,\star} + c_{p,d} \ln(T/\theta_e) - R_d \ln(p_d/\pi), \quad (4.25)$$

$$\mathfrak{s}_w = \mathfrak{s}_{w,\star} + c_w \ln(T/\theta_e), \quad (4.26)$$

fom which it follows that

$$c_e \ln(T/\theta_e) - R_e \ln(p_d/\pi) + q_v L_v + q_v (\mathfrak{s}_v - \mathfrak{s}_s) = 0, \quad (4.27)$$

given that  $R_e = q_d R_d$  is the specific gas constant of the reference state and

$$\begin{aligned} \mathfrak{s}_v - \mathfrak{s}_w &= \mathfrak{s}_v - \mathfrak{s}_s + \mathfrak{s}_s - \mathfrak{s}_w \\ &= \mathfrak{s}_v - \mathfrak{s}_s + (L_v/T). \end{aligned} \quad (4.28)$$

The difference between the vapor and saturation vapor entropy is measured by the relative humidity,  $\mathcal{H} = p_v/p_s$ ,

$$\mathfrak{s}_v - \mathfrak{s}_s = -R_v \left[ \ln \left( \frac{p_v}{p_0} \right) - \ln \left( \frac{p_s}{p_0} \right) \right] = -R_v \ln \mathcal{H}. \quad (4.29)$$

Given Eq. (4.28) and (4.29), Eq. (4.27) can be solved directly for the temperature corresponding to the reference state  $\{\pi, 0, q_t\}$  corresponding to pressure, vapor specific humidity and the specific mass of condensed water, such that

$$\theta_e = T \left( \frac{\pi}{p} \right)^{R_e/c_e} \Omega_e \exp \left( \frac{L_v q_v}{c_e T} \right), \quad (4.30)$$

where

$$\Omega_e = \left[ 1 + \frac{q_v R_v}{R_e} \right]^{R_e/c_e} \mathcal{H}^{-q_v R_v/c_e} \quad (4.31)$$

$\Omega_e$  is a factor that is very near unity, and (because  $q_t \ll 1$ ) depends only very weakly on the thermodynamic state.

The reference state temperature,  $\theta_e$  is frequently used to measure the thermal state of the atmosphere, and is called the equivalent potential temperature—hence the subscript *e*.

In the special case that  $q_t = 0$  the equivalent potential temperature reverts to the more common dry potential temperature,

$$\theta = T \left( \frac{\pi}{p} \right)^{R_d/c_{p,d}} \quad (4.32)$$

Differing only through the contribution of the entropy of moisture to the final reference state temperature.  $\theta_e$  has an advantage over the dry potential temperature in that it remains invariant under phase changes between the liquid and vapor state, hence it proves to be a powerful variable for describing cloud processes.

There is nothing magic about the reference state, it is merely a convention that admits a convenient physical interpretation: the temperature at a near surface atmospheric pressure after all the water in a parcel has been condensed, so that the total water specific humidity is in the form of condensed liquid water. The counterpart to the equivalent potential temperature is the liquid-water potential temperature,

$$\theta_\ell = T \left( \frac{\pi}{p} \right)^{\frac{R_\ell}{c_\ell}} \Omega_\ell \exp \left( -\frac{L_v q_w}{c_\ell T} \right), \quad (4.33)$$

which is the temperature an air parcel would have if were reversibly brought to the  $\{\pi, q_t, 0\}$  reference state, whose specific gas constant is  $R_\ell = q_d R_d + q_t R_v$ .

The entropy temperatures,  $\theta_e$  and  $\theta_\ell$  are conserved under reversible transformations and can serve as thermodynamic coordinates:  $\theta_e$  carries the strong imprint of moisture, and  $\theta_\ell$  is more strongly influenced by temperature perturbations.

### 4.1.8 Static Energies

Temperatures related to the potential temperatures emerge as conserved quantities for adiabatic transformations in a hydrostatic atmosphere. Because atmospheric pressures are very nearly what they would be assuming a hydrostatic balance, increasingly so at larger scales, it is reasonable to substitute  $-d\phi$ , where  $\phi = gz$  is the gravitational potential, for  $\alpha dp$  in the enthalpy form of the first law. With this substitution, and assuming an adiabatic transformation yields

$$0 = d(\mathfrak{h} + \phi), \quad (4.34)$$

i.e.,  $h = \mathfrak{h} + \phi$ , the potential enthalpy, is conserved for adiabatic transformations. The quantity  $h$ , which we called the potential enthalpy (in analogy to the potential temperature) is usually called the static energy, because it measures the sum of the heat content (or enthalpy) and the potential energy of the air parcel, and hence is the energy of a parcel in the absence of kinetic energy. For purely vapor liquid systems,  $h_e$  or  $h_\ell$  from Eqs. (4.13) and (4.14) can be used to describe the enthalpy, from which follows that

$$h_e = c_e T + L_v q_v + \phi, \quad (4.35)$$

and

$$h_\ell = c_\ell T - L_v q_w + \phi, \quad (4.36)$$

are conserved. These quantities are called the moist and liquid-water static energies respectively. In the absence of

water both become identical to the dry static energy,

$$s = c_{p,d}T + gz. \quad (4.37)$$

The names arise historically, a more informative terminology would be to refer to the condensation and evaporation potential temperatures (Eqs. (4.30) and (4.33)) and the condensation and evaporation potential enthalpies (Eqs. (4.35) and (4.36)) respectively.

#### 4.1.9 Further Thoughts on Thermodynamic Variables

Many of the nuances of that moisture brings to thermodynamic descriptions can be neglected when considering small perturbations about a given state. For instance, consider that the liquid water and equivalent enthalpy both describe the enthalpy of a moist liquid-vapor system, hence in general we expect

$$h_e = h_\ell. \quad (4.38)$$

This equality implies that  $(c_e - c_\ell)T = q_t L_v$ . This is of course the case<sup>1</sup> because  $c_\ell - c_w = q_t(c_v - c_w)$  and Kirchoff's relation for the enthalpy of vaporization demands  $(c_v - c_w)dT = dL_v$ . However, if as commonly done one assumes that  $c_\ell \approx c_{p,d} = c_p$  and  $c_e \approx c_{p,d} = c_p$  large differences between  $h_\ell$  and  $h_e$  become apparent. These differences are however differences only in the absolute sense, as defining

$$h_e \approx c_p T + L_v q_v + gz \quad (4.39)$$

$$h_\ell \approx c_p T - L_v q_\ell + gz \quad (4.40)$$

implies that small perturbations (denoted by primes) are proportional, i.e.,  $h'_e \approx h'_\ell$ , where the approximation is exact in the case of saturated perturbations,  $q'_v = q'_s = -q'_\ell$ , even if  $h_\ell$  and  $h_e$  approximated in this fashion differ from one another markedly (proportional to  $q_t L_v$ .) For this reason it is customary to define many of the moist thermodynamic variables in an approximate form appropriate to the consideration of small fluctuations, i.e., the above forms for the moist enthalpy is used to define the static energies, and

$$\theta_e \approx \theta \exp\left(\frac{q_v L_v}{c_p T}\right) \quad \text{and} \quad \theta_\ell \approx \theta \exp\left(\frac{-q_w L_v}{c_p T}\right). \quad (4.41)$$

Such an approach renders many of the distinctions made above, between  $c_e$  or  $c_\ell$  and  $c_{p,v}$  or  $c_w$  mute, their chief purpose being one of pedagogical precision. Many experienced practitioners develop an intuition for which approximations are appropriate, however to the unfamiliar student of the subject the approach to the topic can appear unrigorous and confusing.

<sup>1</sup> Here it is assumed that the specific heats (and hence gas constants) are not allowed to vary with temperature. This is not strictly true. For a perfect gas the specific heats are related to the degrees of freedom of a molecule over which thermal energy is equally distributed. However not all the degrees of freedom of molecules in the atmosphere are accessible at the temperatures found in the atmosphere, and accessibility of additional degrees of freedom increases with temperature, given the gas "constants" a temperature dependence which is neglected in almost all practical applications.

The question often arises as to which thermodynamic coordinates are most appropriate for a given purpose. Enthalpy based variables are well suited to the treatment of mixing processes, because they are so naturally linked to the extensive variables. In contrast the potential temperatures are exponential functions of entropy, namely,

$$s_\ell = s_{\text{cnst}} + \ln \theta_\ell, \quad (4.42)$$

which suggests that the log of the potential temperatures might be a better coordinate for treating, for instance, mixing processes as compared to the potential temperature itself. But because in almost all situations departures from the hydrostatic pressure are of little importance to the dynamics, it is simpler to work directly with the static energies. To the extent approximations such as those given by Eq. (4.40) and (4.39) are acceptable it is also straightforward to derive the enthalpy temperatures,

$$T_e \equiv T + \left(\frac{L_v q_v + gz}{c_{p,d}}\right) \approx \theta + \left(\frac{L_v q_v}{c_{p,d}}\right) \quad (4.43)$$

$$T_\ell \equiv T - \left(\frac{L_v q_\ell + gz}{c_{p,d}}\right) \approx \theta - \left(\frac{L_v q_w}{c_{p,d}}\right) \quad (4.44)$$

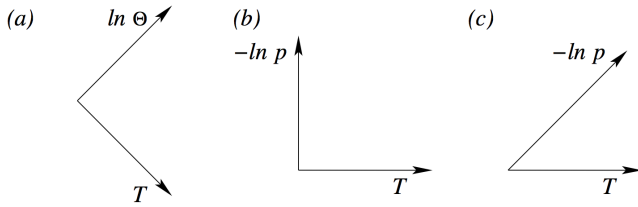
for those that are more comfortable working with temperatures, as compared to enthalpies or entropies.

As for deciding whether the condensed,  $h_e$ , or evaporated  $h_\ell$ , reference state representation is more or less favorable, traditionally systems in which liquid water is explicitly accounted for favor  $h_\ell$  because it reduces to the dry static energy in the absence of condensate. Systems in which water vapor is the other thermodynamic coordinate on the other hand tend to favor  $h_e$  representations, in part because  $h_e$  is, given the approximations in Eq. (4.44) conserved even for open systems in which precipitation converges or diverges from a parcel of air. In the more exact representations of  $h_e$  precipitation affects the value of the total water, and hence  $c_e$ , and thus acts as an enthalpy source. Nonetheless,  $\theta_e$  or  $h_e$  are widely used to describe large-scale systems, particularly insofar as they involve precipitation, and  $\theta_\ell$  or  $h_\ell$  find favor in studies of non-precipitating boundary layer clouds.

## 4.2 Thermodynamic Diagrams

Thermodynamic diagrams can be used to represent the state of a system, as well as thermodynamic processes. They can convey a wide range of information to the trained eye. This information can be critical in determining the subsequent evolution of the atmosphere and is routinely used by weather forecasters to evaluate the likelihood of different events, ranging from fog formation to the development of vigorous deep convective overturning.

Many of these diagrams predate the widespread use of digital computers, and are motivated by the difficulty associated with directly computing various quantities, but even today the diagrams can be useful for visualizing important aspects of the systems state. Hence they continue to be used through the present day. The most common thermodynamic



**Figure 4.2** Thermodynamic coordinates for: (a) tephigram; (b) emagram; (c) skew-emagram ( $\ln P$  Skew- $T$ ).

diagram is the the Skew- $T$  diagram, and its cousin the Tephigram. Other diagrams you might have heard of include the Clapeyron, the Emagram or Neuhoff, and the Stüve diagram (e.g., Fig. 4.2). For the study of moist process it is often useful to represent the plane in terms of moist coordinates, such as  $\theta_e$  and  $q_t$ , for instance as is done in the Paluch diagram.

From the perspective of their thermodynamic coordinates, the Skew- $T$  and the Emagram are identical, they only differ in that the coordinate axes are not orthogonal in the latter. Because isotherms will be parallel to the  $\ln p$  axis and isobars will be parallel to the temperature axis, changing the orientation of the former leads to a greater distinction between isotherms and adiabats. On the Skew- $T$  they are almost perpendicular. In both,  $\ln p$  is used as a thermodynamic coordinate (as opposed to  $p$ ) because the height of an isothermal atmosphere is proportional to  $\ln p$  rather than  $p$ . As a result the vertical coordinate of such a diagram is closely related to the height above the surface. The Tephigram differs from the Emagrams in that pressure is not a coordinate axis. However for the special case of a dry adiabat the first law dictates that

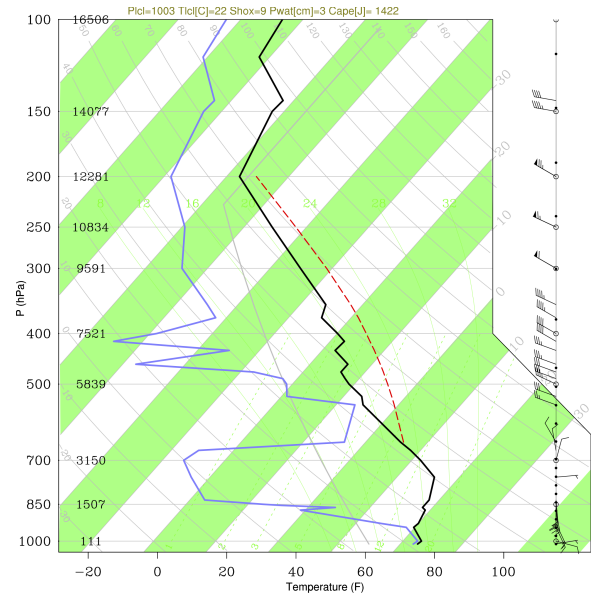
$$\ln \theta - \ln T = -\frac{R_d}{c_p} \ln p + \text{const.} \quad (4.45)$$

Thus associating the  $y$ -axis with  $\ln \theta$  and the  $x$ -axis with  $T$  isobars satisfy  $y = \ln x + C$ . In the range of practical interest the curvature in these lines ends up being small, so that if the diagram is appropriately rotated isobars become approximately horizontal. Thus on a practical level Tephigrams and Skew- $T$ s are very similar, the chief difference being that isobars are curved on the former and adiabats are slightly curved on the latter.

Isobars on a Tephigram and isentropes on a Skew- $T$  are examples of fundamental lines. That is they are not thermodynamic coordinates, but are isolines whose shape is decided by the thermodynamic coordinates. Other examples of fundamental lines include equisaturation curves, pseudo adiabats, and isotherms. Note that by a coordinate transform the fundamental lines of one thermodynamic diagram can serve as the thermodynamic coordinates of another.

Although the great variety of fundamental lines that can be added to a diagram can make it seem rather confusing and busy, they aid the experienced eye in deciding at a glance how the state of the system will change under a given process. To illustrate this consider Fig. 4.3 which presents a standard Skew- $T$  diagram, with the April 16th Brownsville Sounding superimposed.

- Pressure decreases logarithmically upwards, leading to an almost linear relation between height and distance along

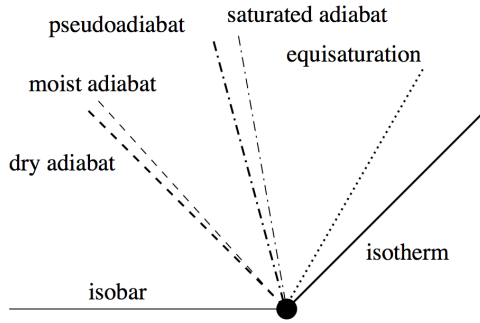


**Figure 4.3** Skew- $T \ln p$  Diagram: (i) pressure decreases logarithmically upwards, leading to an almost linear relation between height and distance along the ordinate — as indicated by the black numbers which denote height in meters just to the right of the pressure values; (ii) isotherms are plotted in  $10^\circ\text{C}$  bands, which are shaded and converted to Fahrenheit where they intersect the abscissa; (iii) adiabats are thin lines with negative slope and slight curvature which intersect the isotherms nearly perpendicularly; (iv) equisaturation curves, are curves of constant specific humidity,  $q_s(p, T)$ , are inclined to the left of the isotherms. Reducing the pressure for a fixed saturation pressure increases the saturation humidity  $q_s$ ; and (v) saturated pseudo-adiabats are almost perpendicular to the abscissa near the surface and curve to become parallel to dry adiabats as  $p$  decreases. Plotted is the Brownsville Texas Sounding for April 16, 2001

the ordinate — as indicated by the black numbers which denote height in meters just to the right of the pressure values.

- Isotherms are plotted in  $10^\circ\text{C}$  bands, which are shaded and converted to Fahrenheit where they intersect the abscissa.
- Adiabats are thin lines with negative slope and slight curvature which intersect the isotherms nearly perpendicularly.
- Equisaturation curves, are curves of constant specific humidity,  $q_s(p, T)$ , are inclined to the left of the isotherms. Reducing the pressure for a fixed saturation pressure increases the saturation humidity  $q_s$ .
- Saturated pseudo-adiabats are almost perpendicular to the abscissa near the surface and curve to become parallel to dry adiabats as  $p$  decreases.

The relative orientation of the fundamental lines on the Skew- $T$  diagram are illustrated more abstractly in Fig. 4.4. Among other things this figure illustrates slight differences in the slope of the dry versus the moist adiabat. Both are given by (4.33), with  $q_\ell = 0$ . The difference arises because for the moist system the heat capacity and specific gas constant account for the presence of water vapor. So doing lowers the



**Figure 4.4** Relative orientation of fundamental lines. Adapted from Fig. VI-4 of Iribarne and Godson.

ratio  $R/c_p$  hence temperature decreases less rapidly with decreasing pressure along a moist adiabat than it does along a dry adiabat. In general the difference between the moist and the dry adiabats will depend on the moisture content of the air. For this reason moist adiabats are not fundamental lines in the plane. They could be represented as fundamental surfaces on three dimensional diagrams, with moisture content marking the third dimension, but this becomes difficult to visualize.

Another interesting way to visualize the atmosphere is in terms of  $\theta_e$  and  $\theta_s$  plotted versus height in the atmosphere. Here we note that the saturation equivalent potential temperature, given as

$$\theta_s \approx \theta \exp\left(\frac{L_v q_s}{c_p T}\right) \quad (4.46)$$

is only a function of temperature and pressure, so in this diagram,  $\theta_s$  measures the thermal structure of the atmosphere, and is constant following a saturated pseudo-adiabat, while the difference between  $\theta_s$  and  $\theta_e$  measures the sub saturation, as  $\theta_e$  is almost always less than  $\theta_r$ .

#### 4.2.1 Pseudo-Adiabats

For the same reason that moist adiabats do not constitute fundamental lines, saturated adiabats, such as lines of constant  $\theta_e$  also are not uniquely representable in the  $p - T$  plane. For this reason the concept of a pseudo-adiabat is often introduced. In a pseudo-adiabatic expansion the contribution of condensed water to the heat capacity of the system is neglected. This is equivalent to assuming that upon condensation any liquid water that forms is precipitated from the system, thus implying an open system which is no longer isentropic. But because differences between this and the reversible process underlying the definition of  $\theta_e$  are small we can consider it as a pseudo-isentropic process, and call the lines corresponding to it pseudo-isentropes, or pseudo adiabats – the latter being more common.

Pseudo-adiabats have the advantage of being representable on the plane. Their disadvantage is that no known closed form expression for their representation exists. The problem arises because the equation for the second law, Eq.(4.15), cannot be directly integrated if the specific heats, and gas constants (which appear in the expression for  $\mathfrak{h}$  and  $\alpha$  respectively) can no longer be assumed to be constant.

This is the case when the total water present is assumed to adjust to the saturation humidity, which is a function of temperature and pressure.

A closed form for the temperature evolution with height can be readily derived from the condition that  $dh_e = 0$ . Starting from (4.35) and noting that  $dh_e$  depends on  $dq_v$ , where for a saturated system  $q_v$  varies as  $q_s$  and  $q_s = f(T, p)$ . Thus through the constraint that  $dh_e = 0$  is straightforward to show that

$$\Gamma_s \equiv -\frac{dT}{dz}\bigg|_{h_s} = \frac{g}{c_p} \left[ \frac{1 + \frac{L_v q_s}{q_d R_d T}}{c_p + \frac{L_v^2 q_s}{R_v c_p T^2} \left( \frac{q_s R_v}{q_d R_d} - 1 \right)} \right] \quad (4.47)$$

with

$$c_p = q_d c_{p,d} + q_v c_{p,v} + q_w c_w \quad (4.48)$$

Note that (4.47) differs from the dry adiabatic lapse rate  $\Gamma_d \equiv g/c_{p,d}$  by the ratio of  $c_{p,d}$  to  $c_p$ , the latter being an effective isobaric specific heat that takes into account the contribution to the enthalpy from the different constituents of the air parcel. Thus given the temperature at any height the temperature at other heights can be calculated by integrating  $\Gamma_s$ . The pseudo-adiabat ( $\Gamma_{\bar{s}}$ ) which neglects the contributions from liquid water to the specific heats can be similarly derived. Doing so leads to a somewhat larger lapse rate, which makes sense because of the higher specific heat capacity of liquid water relative to the other constituents of the air parcel.

A straightforward calculation of the difference between pseudo-adiabats and adiabats shows that they differ most appreciably in the upper troposphere. This is to be expected because  $T$  decreases with height, hence  $q_s$  decreases and  $q_w$  increases correspondingly. Overall the differences in  $T$  tend to be less than 0.5 K below 400 hPa increasing to as much as 3-5K between 100 and 200 hPa. Because in the upper troposphere ice processes play an increasingly important role the lack of correspondence between  $\Gamma_s$  and  $\Gamma_{\bar{s}}$  is less of a concern because the relevance of  $\Gamma_s$  is questionable here in the first place.

#### 4.2.2 Soundings

Soundings in the atmosphere are measurements of its state as a function of altitude. Generally they are made with a balloon which drifts with the mean wind as it rises and measures the wind vector, temperature and relative humidity along its trajectory. An example of such a sounding is shown in Fig. 4.3. Plotted are two lines, the right-most one denoting the temperature structure of the atmosphere, and the leftmost one being the dew-point temperature,  $T_d$ . The dew-point temperature measures the temperature at which air with a given moisture content would saturate, assuming no change in air pressure. Hence it is generally lower than, but independent of the actual temperature. The wind is plotted with wind-barbs on the far-right, wherein the magnitude of the wind is denoted by the barbs and the direction by the orientation. Some things worth noting about the sounding are the following:



- The wind is primarily from the west at upper levels, and reaches a maximum of approximately 75 knots near 200 hPa.
- Surface winds are from the east with a southerly jet at 850 hPa.
- For the most part the temperature profile fluctuates between a rate of increase bounded by  $\Gamma_d$  and  $\Gamma_s$ , tending to the latter between 600 and 200 hPa. Exceptions include a layer between 925 and 750 hPa in which the temperature is more isothermal than moist adiabatic, and a layer above 200 hPa where the profile is again more isothermal.
- The minimum temperature at 200 hPa denotes the tropopause. The warmest air is at the surface, although if the air at 750 hPa were brought to the surface following a dry adiabat, it would be very much warmer, nearly 40 °C.
- If air at the surface could be mechanically forced (along a saturated adiabat) to an altitude of approximately 650 hPa it would begin to be warmer than its environment. Its subsequent trajectory is denoted by the dashed line.
- The moisture structure of the atmosphere is varied. A moist layer can be found at the surface but the warm air just above this is very dry, as is the air in the upper troposphere.

We will return to some of these features when in the context of the discussion of atmospheric stability below

### 4.3 Moist convective instability

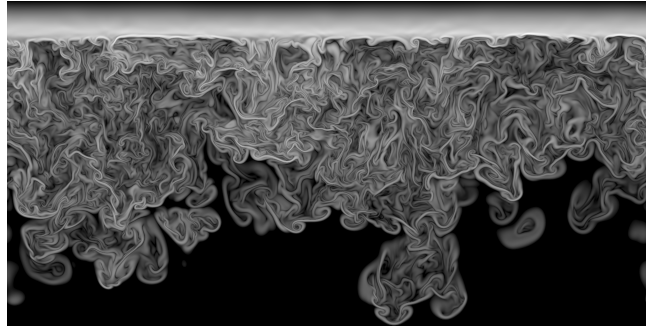
#### 4.3.1 Buoyancy Reversal

Another interesting phenomenon, thought to have broad consequences in the atmosphere is the idea of buoyancy reversal. Here the isobaric mixing of two systems can, through non-linearities in the equation of state, lead to a mixture whose temperature is not bounded by the temperatures of the constituent air masses. To illustrate this possibility we first consider the somewhat simpler situation whereby two air masses can mix to produce a mixed airmass whose temperature is less than either constituent. Physically one can imagine this happening when a saturated airmass (cloud) mixes with a subsaturated (dry) airmass resulting in evaporative cooling which cools the mixture. Clearly the maximum evaporative potential is realized when the mixed parcel is just saturated, implying that all of the liquid water in the cloudy parcel has been evaporated.

Once again, we take as our starting point the two parcels in Fig. ???. The two states are given by  $(h_1, q_{t1})$  and  $(h_2, q_{t2})$  but chosen such that  $q_{t,1} > q_s(T_1, p)$ ,  $q_{t,2} < q_s(T_2, p)$ , and  $T_2 > T_1$ . In words parcel 2 is relatively cool and saturated. Parcel 1 is subsaturated and relatively warm. The moist enthalpy of the system before and after mixing isobarically is the same, that is

$$h_{e,1} + \chi \Delta h_e = h_e. \quad (4.49)$$

which if the mixture remains saturated (and we neglect compositional effects on  $c_p$  and take  $L$  to be independent of



**Figure 4.5** Convective plumes developing as a result of the buoyancy reversal instability

temperature), can be expanded as follows

$$c_p T_1 + L_v q_s(T_1, p) + \chi \Delta h = c_p T + L_v q_s(T, p). \quad (4.50)$$

For small temperature changes, which we denote  $\delta T = T - T_1$ ,

$$q_s(T, p) - q_s(T_1, p) \simeq \left. \frac{\partial q_s}{\partial T} \right|_{T_1} \delta T = \frac{q_s L_v}{R_v T_1^2} \left( \frac{p}{p - p_s(T_1)} \right) \delta T. \quad (4.51)$$

Substituting this expression for  $\delta q_s$  into the previous expression yield a relationship between the perturbation to the temperature under mixing to the differences in the enthalpies of the two states:

$$\left( c_p + \frac{L_v^2 q_s(T_1, p)}{R_v T_1^2} \frac{p}{p - p_s(T_1)} \right) \delta T = \chi \Delta h_e. \quad (4.52)$$

Hence temperature can reverse itself,  $T < T_1 < T_2$  if  $\Delta h < 0$ . The condition for the latter, given that  $\Delta T > 0$ , is approximately that  $-\Delta q > (c_p/L_v)\Delta T$ . In words the warmer parcel must be sufficiently subsaturated to offset its temperature difference.

Note the effective specific heat, given by the prefactor to  $\delta T$  on the lhs of the above, is similar to that which arises due to the phase change effect in the derivation of the moist adiabat given by Eq. 4.47, the difference being in this case the effects of  $p$  on  $q_s$ , which show up in the numerator of (4.47), play role. Not surprisingly this condition  $\Delta h_e < 0$  can also be expressed as the condition on the moist static energy or the equivalent potential, namely that  $\Delta h < 0$  or  $\Delta \theta_e < 0$

The condition (4.52) for temperature reversal can be readily extended to the case of buoyancy reversal by examining under what conditions  $\delta T_\rho < 0$ . In deriving a condition on this, the additional complication comes from the fact that  $T_\rho$  depends on both temperature and moisture. Similarly more exact expressions can be found by restoring the compositional dependencies on the specific heats, and accounting for the temperature dependence of  $L_v$  given by Kirchoff's relation.

#### 4.3.2 Conditional Instability

Except for the few tens of meters nearest the ground during a warm day, the oscillation frequency of gravity waves in an





energy. That is it bounds the amount of kinetic energy a parcel could have, thus defining a velocity scale:

$$w_{\max} = \sqrt{2\mathcal{A}} \quad (4.57)$$

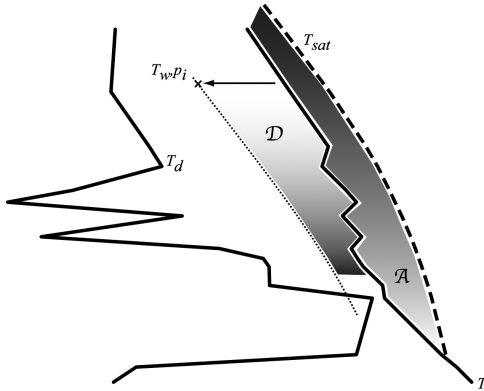
which is another measure of the intensity of convection.

#### 4.3.3.1 Other CAPE-like Measures

CAPE is the most common measure of the potential instability of the atmospheric column. While it constitutes a necessary condition for instability, many other factors can come into play. The amount of work that must be invested to access the CAPE of a sounding is highly variable. In some cases one does not have to do a large amount of work on a parcel before the atmosphere starts returning the favor. To measure this aspect of the atmosphere another parameter, called Convective Inhibition (or CIN) is sometimes introduced. It is the analog to CAPE, but measures the amount of work that must be done to lift a parcel from some reference level,  $p_*$ , to its level of free convection:

$$\mathcal{I} = - \int_{p_*}^{p_f} R_d T'_\rho d(\ln p). \quad (4.58)$$

Thus the potential instability of the atmosphere depends not on CAPE alone, but also on other factors such as CIN.



**Figure 4.7** Illustration of different types of CAPE using the Brownsville sounding of Fig. 4.3

Another CAPE like measure of the atmosphere is called down-draft CAPE, or  $\mathcal{D}$ . It is illustrated in reference to regular CAPE in Fig. 4.7.  $\mathcal{D}$  measures the stability of saturated downward displacements of air first brought to its wet-bulb temperature by evaporation of falling rain. The wet-bulb temperature (sometimes denoted  $T_w$ ) is bounded by the dew-point temperature and the actual temperature. It is the temperature that one gets by isobarically bringing air to saturation through evaporation. Because water is evaporated into the air, the air both cools and moistens increasing its mixing ratio while decreasing its temperature. Hence,

$$\mathcal{D}(p_i) = \int_{p_{\text{Evap}}}^{p_{\text{sf}}} R_d T'_\rho (\ln p), \quad (4.59)$$

where here  $T_\rho(p_{\text{Evap}}) = T_w$  and  $T'_\rho$  is the difference between the virtual temperature of a parcel at some level  $p_{\text{eval}}$

brought to saturation by evaporating water into it and the environmental value of  $T_\rho$ . Physically  $\mathcal{D}$  measures the stability of air to evaporation of rain. In environments with large values of  $\mathcal{D}$  vigorous down-drafts can be formed by evaporating water (from precipitation) into a dry ambient environment. The analogy to buoyancy reversal, whereby negatively buoyant parcels can be created through isobaric mixing, should be readily apparent.

#### 4.3.3.2 Caveats on CAPE

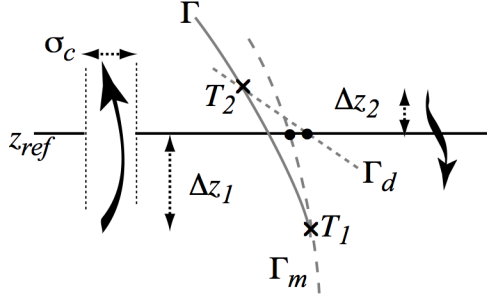
As alluded to above the actual CAPE of a given atmospheric sounding is not a number without ambiguity. This ambiguity stems mostly from the varied definitions associated with it. Above CAPE has been defined to be the positive area on the thermodynamic diagram. Others define it to be the *net* positive area associated with a parcel lifted dry adiabatically from the surface to the level of neutral buoyancy. Others compute the CAPE associated with a parcel characterized by the *mean* properties of the lower 10-50 hPa of the atmosphere. Yet others adjust the surface properties of the sounding to reflect what they anticipate conditions will be like at the height of the day. Because it measures a conditional process, there is no way to define CAPE unambiguously, as how much energy is available to a parcel raised to its level of free condition depends very much on the parcel being lifted.

The thermodynamic processes which govern the evolution of the state of the parcel above the lifting condensation level also play a large role in determining CAPE. For instance parcels in which ice forms will have different values of CAPE than parcels assumed to follow a pseudo-adiabat, likewise rising parcels usually mix with their environment, intact the more unstable they are the more they are likely to mix. Thus CAPE can be seen as a function of the state of a parcel, the environment, and the specified type of thermodynamic process for the rising parcel. These ambiguities do not diminish the value of a measure like CAPE, but they do indicate that if it is to be used quantitatively the particular use of the concept must be made precise.

#### 4.3.4 Slice Method

The ability of CAPE to characterize the potential energy available to convection is based on a number of idealizations. These include: (i) that the parcels being lifted follow the specified thermodynamic process; (ii) that the parcels being lifted are characteristic of the air that actually fuels any convection that develops; (iii) that the response of the environment can be neglected. More refined measures of convective instability attempt to address one or more of these limitations. Most notable among these is the slice method introduced by Bjerknes. In some sense it can be viewed as a theory for fully developed convection.

In the slice method we evaluate the situation in a convecting atmosphere at some reference height  $z_*$  above cloud base. It is envisioned that in some sufficiently large area, the fraction of the convecting region is a robust quantity which can be denoted by  $\sigma_c$ . If we integrate the vertical velocity



**Figure 4.8** Situation modeled by the slice method. Here the depicted convection is energetically inhibited because the response of the environment would stabilize the atmosphere.

over the convecting region we can define a mean convective velocity  $w_c$ . Mass continuity across the reference level allows one to express the mean subsiding velocity of the environment  $\tilde{w}$  in terms of  $(\sigma_c, w_c)$  :

$$\sigma_c w_c + (1 - \sigma_c) w = 0, \quad (4.60)$$

where symbols without a subscript denote environmental values, so here  $w$  is the mean vertical velocity of the environment.

Assuming that the air rising in the convecting region is rising along a saturated adiabat, while the subsiding air follows a dry adiabat, the temperature difference between the convecting region and the environment at the reference level can be expressed as follows:

$$T_c(z_*) - T(z_*) \approx T_{01} - \Gamma_s \Delta z_1 - (T_{02} - \Gamma_d \Delta z_2), \quad (4.61)$$

where  $\Delta z_2 < 0 < \Delta z_1$ ,

$$T_{01} = T(z_*) + \Delta z_1 \Gamma \quad (4.62)$$

$$T_{02} = T(z_*) + \Delta z_2 \Gamma, \quad (4.63)$$

and  $\Gamma = -dT/dz$  the environmental lapse rate. Note that the above implies that our model is most appropriate if we associate  $\Delta z_1$  with the distance above cloud base. The above relations are illustrated in schematically Fig. ??, they describe the temperature in the convecting region as the temperature that the environmental air a distance  $\Delta z_1$  below  $z_*$  would have it if were lifted to  $z_*$  along a moist adiabat, while the temperature of the environment at  $z_*$  is that which environmental air a distance  $-\Delta z_2$  above the convecting region would have were it brought dry adiabatically to  $z_*$ . The distances  $\Delta z_1$  and  $\Delta z_2$  are given by  $w_c \Delta t$  and  $\tilde{w} \Delta t$  respectively.

Substituting from above allows us to express the temperature difference in terms of  $\sigma_c, w_c$  and  $\Gamma$ ,

$$\frac{T_c - \tilde{T}}{\Delta t} = w_c (\Gamma - \Gamma_s) - \tilde{w} (\Gamma - \Gamma_d) \quad (4.64)$$

$$= w_c \left[ \Gamma - \Gamma_s + \frac{\sigma_c}{1 - \sigma_c} (\Gamma - \Gamma_d) \right]. \quad (4.65)$$

Thus the criterion for convective instability,  $T_c - \tilde{T} > 0$  is equivalent to requiring that

$$\Gamma > \Gamma_s + \sigma_c (\Gamma_d - \Gamma_s) > \Gamma_s. \quad (4.66)$$

This requirement is more severe than that given by the parcel method. The physical difference being that the compensating downward motion stabilizes the environment.

A further implication of the above result is that moist convection is most unstable if its fractional area is smallest. Physically we might expect this, as for a given  $w_c$  vanishing  $\sigma_c$  implies that the compensating environmental motion is minimized and hence the environment is stabilized the least. Mathematically we can see this by considering the neutral limit of (4.65):

$$\Gamma - \Gamma_s + \frac{\sigma_c}{1 - \sigma_c} (\Gamma_d - \Gamma) = 0. \quad (4.67)$$

which can be solved for  $\sigma_c$  :

$$\sigma_c \leq \frac{\Gamma - \Gamma_s}{\Gamma_d - \Gamma_s}. \quad (4.68)$$

#### 4.4 Fluid Dynamical Systems

From the perspective of a rotating coordinate system in a gravitational field the governing equations of a two component fluid for which viscous dissipation is not modified by compressibility effects are:

$$\frac{D}{Dt} u_i = -g \delta_{i3} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (4.69)$$

$$\frac{D}{Dt} \rho = -\rho \frac{\partial u_j}{\partial x_j} + \dot{\rho}_c \quad (4.70)$$

$$\frac{D}{Dt} \mathbf{s} = \frac{\mathbf{q}}{T} \quad (4.71)$$

$$\frac{D}{Dt} q_t = \mathbf{m}_t. \quad (4.72)$$

Tensor notation is employed so that  $u_i$  with  $i \in \{1, 2, 3\}$  is the velocity vector referred to a Cartesian coordinate system  $x_i \in \{x, y, z\}$  such that  $x_3 \equiv z$  is aligned with the gravitational acceleration vector. Throughout we use Einstein's summation convention with  $\delta_{ij}$  Kroenecker's delta. The derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j},$$

is the convective derivative, it denotes a change in the fluid property following the flow. The kinematic viscosity is denoted by  $\nu$ , the thermal diffusivity by  $\kappa$ ,  $f$  is the Coriolis parameter and  $\mathfrak{S}$  is an entropy source, and  $\mathcal{M}_t$  is a water mass source. for instance radiation or precipitin, the remaining symbols retain their conventional meanings.

Equations (4.69)-(4.72) follow directly from the laws of classical mechanics and thermodynamics, except for Eq. (4.72), which is formed by decomposing the mass continuity equation into several equations, one for each constituent, and noting that  $\rho_t = q_t \rho$ . Subtracting the equation for  $q_t D\rho/Dt$  from the equation for  $D(\rho q_t)/Dt$  yields Eq. (4.72). For an equilibrium system the water mass is partitioned among the phases so as to maintain saturation with respect to ice, or water. For non-equilibrium phases of water, for instance precipitation, the system must be augmented to incorporate additional mass conservation equations, exchange rates between the phases, and associated entropy

sources. Because  $\mathfrak{s} = \mathfrak{s}(p, T, q_t)$  Eqs. (4.69)-(4.72), form a closed system given a specification of the source terms,  $(\mathfrak{S}, \mathcal{M}_t)$ , and external parameters  $(g, \nu)$ , and an equation of state.

#### 4.4.1 Boussinesq equations

It turns out that the above set of equations are rarely used as a basis for the investigation of fluid processes in the meteorology or oceanography. Their chief shortcoming is that they include physical processes, such as sound waves, whose timescales are much shorter than the timescales of most processes of interest. Retaining these modes of variability maintains needless complexity and hinders theoretical studies.

Acoustic modes can be limited by introducing a constraint on compressibility. A number of approximate equation sets, based on such a constraint, have been proposed and are used. All preserve what are thought to be the essential aspects of the fluid dynamics, while filtering sound waves. These simplified systems can be grouped into three categories based on what continuity equation they imply. Here the procedure is illustrated by sketching the derivation for the Boussinesq equations for a moist atmospheric flow. These equations are asymptotically exact in the limit of small fluctuations in thermodynamic quantities. For a flow incorporating phase changes it proves useful to work with  $(p, \theta_\ell, q_t)$  as the thermodynamic coordinates. Expanding these coordinates in a base state and fluctuating quantities as follows

$$p = p_0 + \bar{p}(z) + p'(x, y, z, t) \quad (4.73)$$

$$\rho = \rho_0 + \rho'(x, y, z, t) \quad (4.74)$$

$$\theta_\ell = \theta_0 + \theta'_\ell(x, y, z, t) \quad (4.75)$$

$$q_t = q'_t(x, y, z, t), \quad (4.76)$$

where the base state for  $q_t$  is assumed to be zero, i.e., fluctuations in  $q_t$  are not assumed to be small relative to the base state, and  $p_0$  is taken to be 100 000 hPa. Thus for consistency  $\theta_{\ell,0} = \theta_0$ . In the limit of small perturbations

$$\rho(p, T_\rho) \approx \rho_0 + \left(\frac{\partial \rho}{\partial p}\right)_0 p' + \left(\frac{\partial \rho}{\partial T_\rho}\right)_0 T'_\rho \quad (4.77)$$

so that

$$\frac{\rho'}{\rho_0} = \frac{p'}{p_0} - \frac{T'_\rho}{T_{\rho,0}} \quad (4.78)$$

The principle assumptions of the Boussinesq system is that density perturbations are, relative to the mean state, small, such that

$$\varepsilon_\rho \equiv \frac{\rho'}{\rho_0} \ll 1. \quad (4.79)$$

and that pressure perturbations are much smaller than temperature perturbations, so that

$$\frac{\rho'}{\rho_0} = -\frac{T'_\rho}{T_{\rho,0}}, \quad (4.80)$$

as was asserted in the discussion of the Equation of state. This is a good assumption because the pressure is an integral quantity, in that through the hydrostatic balance it depends

on the integral of the temperature field through the atmosphere. So local temperature perturbations are expected to be much larger than pressure perturbations.

Consider the case when the flow is governed by a single velocity-scale  $U$  and length-scale  $H$  (which imply a time-scale  $H/U$ ) determined by the inertial motions. In this case the continuity equation (4.70) can be non-dimensionalized by the time, velocity and distance, scales (with non-dimensional quantities denoted by tilde),

$$(\tilde{t}, \tilde{u}_i, \tilde{x}_i) \quad \text{where} \quad t \rightarrow H\tilde{t}/U, \quad u_i \rightarrow \tilde{u}_i U, \quad x_i = H\tilde{x}_i,$$

such that

$$\frac{D}{D\tilde{t}}(\varepsilon_\rho) + (1 + \varepsilon_\rho) \frac{\partial \tilde{u}_i}{\partial \tilde{x}_i} = 0. \quad (4.81)$$

By (4.79) the leading order balance is simply non-divergence, which in terms of dimensional variables becomes:

$$\rho_0 \frac{\partial u_i}{\partial x_i} = 0. \quad (4.82)$$

To analyze the balances in the momentum equation we multiply (4.69) by  $\rho$ , and follow a similar procedure, whereby we find that

$$(1 + \varepsilon_\rho) \frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{1}{\rho_0 U^2} \frac{\partial(\bar{p} + p')}{\partial \tilde{x}_i} - (1 + \varepsilon_\rho) \frac{H}{U} \left[ \frac{g\delta_{i3}}{U} \right] + \frac{1}{\mathcal{R}} \left[ \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j \partial \tilde{x}_j} \right], \quad (4.83)$$

where  $\mathcal{R} = \mathcal{U}H/\nu$  is the Reynolds number. For the basic state to be static requires that  $d\bar{p}/dz = -\rho_0 g$ . Subtracting this balance from above, substituting from  $\varepsilon_\rho$  from Eqs. (4.80), and returning the equations to their dimensional form yields the asymptotically valid form of the momentum equation,

$$\frac{Du_i}{Dt} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x_i} + \frac{T'_\rho}{T_{\rho,0}} g\delta_{i3} + \nu_0 \frac{\partial^2 u_i}{\partial x_j \partial x_j}. \quad (4.84)$$

The form of this equation makes clear the manner in which fluctuations in the density temperature,  $T_\rho$ , drive density fluctuations. The term  $(T'_\rho/T_{\rho,0})g$  has units of acceleration, it is a reduced gravity and is called the buoyancy term – sometimes denoted by  $b$ .

The thermodynamic equation, (4.71), when written in terms of  $\theta_\ell$  completes the description of the system, yielding the moist Boussinesq equations:

$$\frac{D}{Dt} u_i = -b\delta_{i3} - \frac{1}{\rho_0} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (4.85)$$

$$\frac{D}{Dt} \theta_\ell = \frac{\theta_0}{T} \mathfrak{S} \quad (4.86)$$

$$\frac{D}{Dt} q_t = \mathcal{M}_t, \quad (4.87)$$

subject to the constrain that the flow is non-divergent, and supplemented by the equation of state, which is specified through the buoyancy function,  $b(\theta_\ell, q_t)$ .

#### 4.4.2 Primitive Equations

Here I plan to simply state the equations and the assumptions that lead to them, pointing out two things: (i) working

in pressure coordinates gives a non-divergent constraint; (ii) that the static energies arise as natural coordinates.

#### 4.4.3 Reynolds Averaged Equations

Here I plan to introduce the closure problem, using the primitive equations.

#### 4.4.4 Similarity Approaches

Not sure if I will say anything here, it depends on other chapters. But I have material that establishes similarity as non-dimensional equivalence, and gives some nice examples, i.e., the period of a pendulum, as an introduction to parameterization.

### 4.5 Summary of Symbols

#### Exercises

1. Prove that  $L_v/T = s_v - s_w$  in the case of a saturated system.
2. Derive  $\gamma_\ell$  and  $\Omega_\ell$ , which appear in Eq. (??) which defines the liquid-water potential temperature.
3. Show that

$$N_s^2 \approx \frac{g}{\theta} \frac{d\theta_e}{dz}$$

[hp]

Table 4.2. *Summary of symbols*

Quantity	Symbol	Reference
Relative Humidity	$\mathcal{H}$	
Temperature	$T$	
Density	$T_\rho$	
Air pressure	$p$	
Partial pressure of dry air	$p_d$	
Vapor pressure	$p_v$	
Saturation vapor pressure	$p_s$	
Specific humidity (vapor)	$q_v$	
Specific humidity (liquid condensate)	$q_w$	
Specific humidity (solid/ice condensate)	$q_i$	
Total water specific humidity	$q_t$	
Total condensate specific humidity	$q_c$	
Potential Temperature	$\theta$	
Equivalent Potential Temperature	$\theta_e$	(4.30)
Liquid-water Potential Temperature	$\theta_\ell$	(??)
Saturation Equivalent Potential Temperature	$\theta_s$	
Liquid-water Potential Density Temperature	$\theta_{\rho,\ell}$	
Dry static energy	$s$	
x Moist static energy	$h_e$	(4.30)
Liquid-water static energy	$h_\ell$	(??)
Saturation Equivalent static energy	$h_s$	
Liquid-water density static energy	$h_{\rho,\ell}$	
Reference pressure	$\pi = 10^5 \text{ Pa}$	
Density	$T_\rho$	
Entropy	$\mathfrak{s}$	
Enthalpy	$\mathfrak{h}$	
Gibbs Free Energy	$\mathfrak{g}$	
Heating	$\mathfrak{q}$	
Brunt-Väisällä frequency	$N^2$	
Mixing fraction	$\chi$	
CAPE	$\mathcal{A}$	
Convective Inhibition	$\mathcal{I}$	
Downdraft CAPE	$\mathcal{D}$	
Specific gas constant	$R$	
Isobaric specific heat	$c_p$	
Velocity vector	$u_i = \{u, v, w\}$	
Temperature lapse rate	$\Gamma$	
Convective updraft area	$\sigma_c$	
Reynolds Number	$\mathcal{R}$	

Any notes... Subscript notation?